should seem believable to you that any reasonable subset of $\mathbf{R}^N$ whose horizontal slices all have $\mathbf{R}^k$-measure zero must itself have measure zero in $\mathbf{R}^N$; this is a form of "Fubini's theorem," proved in Appendix A. Granting it for the moment, we conclude that every S_α has measure zero in $\mathbf{R}^N$.

Obviously, a function has a degenerate critical point on X if and only if it has one in some U_α. Thus the set of N-tuples a for which f_a is not a Morse function on X is the union of the S_α. Since a countable union of sets of measure zero still has measure zero, we are finished. Q.E.D.

EXERCISES

1. Show that $\mathbf{R}^k$ is of measure zero in $\mathbf{R}^l$, $k < l$.

2. Let A be a measure zero subset of $\mathbf{R}^k$. Show that $A \times \mathbf{R}^l$ is of measure zero in $\mathbf{R}^{k+l}$. (This implies Exercise 1. How?)

3. Suppose that Z is a submanifold of X with $\dim Z < \dim X$. Prove that Z has measure zero in X (without using Sard!).

4. Prove that the rational numbers have measure zero in $\mathbf{R}^1$, even though they are dense.

5. Exhibit a smooth map $f: \mathbf{R} \longrightarrow \mathbf{R}$ whose set of critical values is dense. [HINT: Write the rationals in a sequence $r_0, r_1, \ldots$. Now construct a smooth function on $[i, i+1]$ that is zero near the endpoints and that has r_i as a critical value (Figure 1-26).]

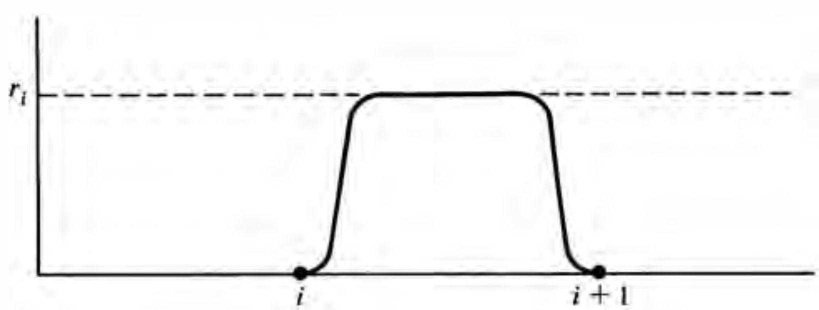

Figure 1-26

*6. Prove that the sphere S^k is simple connected if $k > 1$.
[HINT: If $f: S^1 \longrightarrow S^k$ and $k > 1$, Sard gives you a point $p \notin f(S^1)$. Now use stereographic projection.]

7. When $\dim X < \dim Y$, Sard says that the image of any smooth map $f: X \longrightarrow Y$ has measure zero in Y. Prove this "mini-Sard" yourself, assuming the fact that if A has measure zero in $\mathbf{R}^l$ and $g: \mathbf{R}^l \longrightarrow \mathbf{R}^l$ is

smooth, then $g(A)$ also has measure zero. [HINT: Reduce to the case of a map f of an open set $U \subset \mathbf{R}^k$ into $\mathbf{R}^l$. Consider $F: U \times \mathbf{R}^{l-k} \longrightarrow \mathbf{R}^l$ defined by $F(x, t) = f(x)$.]

8. Analyze the critical behavior of the following functions at the origin. Is the critical point nondegenerate? Is it isolated? Is it a local maximum or minimum?
(a) $f(x, y) = x^2 + 4y^3$
(b) $f(x, y) = x^2 - 2xy + y^2$
(c) $f(x, y) = x^2 + y^4$
(d) $f(x, y) = x^2 + 11xy + y^2/2 + x^6$
(e) $f(x, y) = 10xy + y^2 + 75y^3$

9. Prove the Morse Lemma in $\mathbf{R}^1$. [HINT: Use this elementary calculus lemma: for any function f on R and any point $a \in R$, there is another function g such that

$$f(x) = f(a) + (x - a)f'(a) + (x - a)^2 g(x).$$

This result is proven on page 135.]

10. Suppose that $f = \sum a_{ij}x_ix_j$ in $\mathbf{R}^k$. Check that its Hessian matrix is $H = (a_{ij})$. Considering $\mathbf{R}^k$ as the vector space of column vectors, H operates as a linear map by left multiplication, as usual. Show that if $Hv = 0$, then f is critical all along the line through v and 0. Thus the origin is an isolated critical point if and only if H is nonsingular.

11. Using the Morse Lemma, prove that if a is a nondegenerate critical point of a function f, there exists a local coordinate system $(x_1, \ldots, x_n)$ around a such that

$$f = f(a) + \sum_{i=1}^{n} \epsilon_i x_i^2, \qquad \epsilon_i = \pm 1.$$

[HINT: Diagonalize (h_{ij}).]

12. Prove that the function f in Exercise 11 has a maximum at a if all the ϵ_i's are negative and a minimum if they are all positive. Show that if neither is the case, then a is neither a maximum nor a minimum.

13. Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.

14. Show that the "height function" $(x_1, \ldots, x_k) \longrightarrow x_k$ on the sphere S^{k-1} is a Morse function with two critical points, the poles. Note that one pole is a maximum and the other a minimum.

15. Let X be a submanifold of $\mathbf{R}^N$. Prove that there exists a linear map $l: \mathbf{R}^N \longrightarrow \mathbf{R}$ whose restriction to X is a Morse function. (Exercise 14 is a special case.)

16. Let f be a smooth function on an open set $U \subset \mathbf{R}^k$. For each $x \in U$, let $H(x)$ be the Hessian matrix of f, whether x is critical or not. Prove that f is Morse if and only if

$$\det(H)^2 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}\right)^2 > 0 \qquad \text{on } U.$$

17. Suppose that f_t is a homotopic family of functions on $\mathbf{R}^k$. Show that if f_0 is Morse in some neighborhood of a compact set K, then so is every f_t for t sufficiently small. [Hint: Show that the sum in Exercise 16 is bounded away from 0 on a neighborhood of K as long as t is small.]

18. (Stability of Morse Functions) Let f be a Morse function on the compact manifold X, and let f_t be a homotopic family of functions with $f_0 = f$. Show that each f_t is Morse if t is sufficiently small. [Hint: Exercise 17.]

19. Let X be a compact manifold. Prove that there exist Morse functions on X which take distinct values at distinct critical points. [Hint: Let f be Morse, and let $x_1, \ldots, x_N$ be its critical points. Let ρ_i be a smooth function on X that is one on a small neighborhood of x_i and zero outside a slightly larger neighborhood. Choose numbers $a_1, \ldots, a_N$ such that

$$f(x_i) + a_i \neq f(x_j) + a_j \qquad \text{if} \quad i \neq j.$$

Prove that if the a_i are small enough, then

$$f + \sum_{i=1}^{N} a_i \rho_i$$

has the same critical points as f and is even arbitrarily close to f.]

20. (a) Suppose that X is a compact manifold in $\mathbf{R}^N$ and f is a function on X. Show that the N-tuples $(a_1, \ldots, a_N)$ for which

$$f_a = f + a_1 x_1 + \cdots + a_N x_N$$

is a Morse function constitute an open set.

(b) Remove the compactness assumption on X, and show that the set $\{a : f_a \text{ Morse}\}$ is a countable intersection of open sets. [Hint: Use (a), plus the second axiom of countability.]

(c) The set $\{a : f_a \text{ not Morse}\}$ is therefore a countable union of closed sets. Show that this is enough to justify the use of Fubini in our proof of the existence of Morse functions. (See Appendix A.)

21. Let $\phi: X \longrightarrow \mathbf{R}^N$ be an immersion. Show for "almost every" $a_1, \ldots, a_N$, $a_1\phi_1 + \cdots + a_N\phi_N$ is a Morse function on X where $\phi_1, \ldots, \phi_N$ are the coordinate functions of ϕ. [HINT: Show that the proof we gave for the existence of Morse functions only requires X to be immersed in $\mathbf{R}^N$, not embedded.]

22. Here is an application of Morse theory to electrostatics. Let $x_1, \ldots, x_4$ be points in general position in $\mathbf{R}^3$ (that is, they don't all lie in a plane.) Let $q_1, \ldots, q_4$ be electric charges placed at these points. The potential function of the resulting electric field is

$$V_q = \frac{q_1}{r_1} + \cdots + \frac{q_4}{r_4}$$

where $r_1 = |x - x_i|$. The critical points of V_q are called *equilibrium points* of the electric field, and an equilibrium point is *non-degenerate* if the critical point is. Prove that for "almost every" q the equilibrium points of V_q are non-degenerate and finite in number. [HINT: Show that the map: $\mathbf{R}^3 - \{x_1, \ldots, x_4\} \longrightarrow \mathbf{R}^4$ with coordinates r_1, r_2, r_3, r_4 is an immersion and apply Exercise 21.]

§8 Embedding Manifolds in Euclidean Space

The second application we shall give for Sard's theorem is a proof of the Whitney embedding theorem. A k-dimensional manifold X has been defined as a subset of some Euclidean space $\mathbf{R}^N$ that may be enormous compared to X. This ambient Euclidean space is rather arbitrary when we consider the manifold X as an abstract object. For example, if $M > N$, then $\mathbf{R}^N$ naturally embeds in $\mathbf{R}^M$, so one might have constructed the same manifold X in $\mathbf{R}^M$ instead. Whitney inquired how large N must be in order that $\mathbf{R}^N$ contain a diffeomorphic copy of every k-dimensional manifold. His preliminary answer was that $N = 2k + 1$ suffices; this is the result we shall prove. After a great deal of hard work, Whitney improved his result by one, establishing that every k-dimensional manifold actually embeds in $\mathbf{R}^{2k}$.

One way to interpret the Whitney theorem is as a limit to the possible complexity of manifolds. Any manifold that may be defined in $\mathbf{R}^N$ may also be defined in $\mathbf{R}^{N+1}$; but perhaps the extra room for twisting in $\mathbf{R}^{N+1}$ allows the construction of manifolds there that cannot exist inside the smaller space $\mathbf{R}^N$. (In fact, it is not a priori obvious that any single Euclidean space is large enough to contain all manifolds of a given dimension.) A classic example is the Klein bottle, a surface that can be constructed in $\mathbf{R}^4$ by attaching the